



11.1 The properties of the adjoint

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Introduction

- In this topic, we will
 - Deduce a number of properties of the adjoint
 - Observe that these are similar to the properties of the complex conjugate





Equality of linear maps

- Recall that given two linear maps $A, B: \mathcal{U} \rightarrow \mathcal{V}$, if

$$\langle A\mathbf{u}, \mathbf{v} \rangle = \langle B\mathbf{u}, \mathbf{v} \rangle$$

for all vectors $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v} \in \mathcal{V}$, then $A = B$.





The adjoint of the adjoint

Theorem

$(A^*)^* = A$, that is, the adjoint of the adjoint of a linear map is the map itself.

Proof:
$$\begin{aligned}\langle A\mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{u}, A^* \mathbf{v} \rangle \\ &= \langle A^* \mathbf{v}, \mathbf{u} \rangle^* \\ &= \langle \mathbf{v}, (A^*)^* \mathbf{u} \rangle^* \\ &= \langle (A^*)^* \mathbf{u}, \mathbf{v} \rangle\end{aligned}$$

Therefore, $A = (A^*)^*$ ■





The adjoint of a linear combination

Theorem

$$(\alpha A_1 + \beta A_2)^* = \alpha^* A_1^* + \beta^* A_2^*$$

Proof:

$$\begin{aligned}\langle \mathbf{u}, (\alpha A_1 + \beta A_2)^* \mathbf{v} \rangle &= \langle (\alpha A_1 + \beta A_2) \mathbf{u}, \mathbf{v} \rangle \\ &= \alpha^* \langle A_1 \mathbf{u}, \mathbf{v} \rangle + \beta^* \langle A_2 \mathbf{u}, \mathbf{v} \rangle \\ &= \alpha^* \langle \mathbf{u}, A_1^* \mathbf{v} \rangle + \beta^* \langle \mathbf{u}, A_2^* \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \alpha^* A_1^* \mathbf{v} \rangle + \langle \mathbf{u}, \beta^* A_2^* \mathbf{v} \rangle \\ &= \langle \mathbf{u}, (\alpha^* A_1^* + \beta^* A_2^*) \mathbf{v} \rangle\end{aligned}$$

Therefore, $(\alpha A_1 + \beta A_2)^* = \alpha^* A_1^* + \beta^* A_2^*$ ■





The adjoint is a linear for real linear maps

Theorem

For real vector spaces, the adjoint is a linear.

Proof:

For real numbers, the inner product is symmetric, so we may modify the previous proof to show that

$$(\alpha A_1 + \beta A_2)^* = \alpha A_1^* + \beta A_2^* \quad \blacksquare$$





The adjoint of a composition

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{W}$, then $(BA)^* = A^*B^*$.

Proof:

$$\begin{aligned}\langle \mathbf{u}, (BA)^* \mathbf{w} \rangle &= \langle (BA) \mathbf{u}, \mathbf{w} \rangle \\ &= \langle B(A\mathbf{u}), \mathbf{w} \rangle \\ &= \langle A\mathbf{u}, B^* \mathbf{w} \rangle \\ &= \langle \mathbf{u}, A^* (B^* \mathbf{w}) \rangle \\ &= \langle \mathbf{u}, (A^* B^*) \mathbf{w} \rangle\end{aligned}$$

Therefore, $(BA)^* = A^*B^*$. ■





The adjoint of a composition

- Note that this makes sense:

If $A: \mathcal{U} \rightarrow \mathcal{V}$ and $B: \mathcal{V} \rightarrow \mathcal{W}$, then $BA: \mathcal{U} \rightarrow \mathcal{W}$.

If $A^*: \mathcal{V} \rightarrow \mathcal{U}$ and $B^*: \mathcal{W} \rightarrow \mathcal{V}$, then $A^*B^*: \mathcal{W} \rightarrow \mathcal{U}$,
however, from the first, $(BA)^*: \mathcal{W} \rightarrow \mathcal{U}$.





The adjoint of the zero map

Theorem

If $O: \mathcal{U} \rightarrow \mathcal{V}$ and $O': \mathcal{V} \rightarrow \mathcal{U}$ then $O^* = O'$.

Proof:

$$\begin{aligned}\langle \mathbf{u}, O^* \mathbf{v} \rangle &= \langle O \mathbf{u}, \mathbf{v} \rangle \\ &= \langle \mathbf{0}, \mathbf{v} \rangle = 0\end{aligned}$$

But $\langle \mathbf{u}, O' \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$,

Therefore, $O^* = O'$. ■





The adjoint of the identity operator

Theorem

For $I: \mathcal{U} \rightarrow \mathcal{U}$, then $I^* = I$.

Proof:

$$\begin{aligned}\langle \mathbf{u}_1, I^* \mathbf{u}_2 \rangle &= \langle I \mathbf{u}_1, \mathbf{u}_2 \rangle \\ &= \langle \mathbf{u}_1, \mathbf{u}_2 \rangle \\ &= \langle \mathbf{u}_1, I \mathbf{u}_2 \rangle\end{aligned}$$

Therefore, $I^* = I$. ■





The inverse of the adjoint

Theorem

If $A: \mathcal{U} \rightarrow \mathcal{U}$ and A is invertible,
then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proof:

$$I = A^{-1}A$$

$$I^* = (A^{-1}A)^*$$

$$I = A^* (A^{-1})^*$$

Therefore, $(A^*)^{-1} = (A^{-1})^*$. ■





Properties of the adjoint

- Notice that all these properties were derived just from the definition...
 - We don't even know what the adjoint of a matrix is
- Never-the-less, we can see that the adjoint is similar to the complex conjugate:

$$(z^*)^* = z$$

$$(A^*)^* = A$$

$$(w + z)^* = w^* + z^*$$

$$(A_1 + A_2)^* = A_1^* + A_2^*$$

$$(w - z)^* = w^* - z^*$$

$$(A_1 - A_2)^* = A_1^* - A_2^*$$

$$(wz)^* = w^* z^*$$

$$(BA)^* = A^* B^*$$

- For operators: $(z^{-1})^* = (z^*)^{-1}$

$$(A^{-1})^* = (A^*)^{-1}$$

$$(z^n)^* = (z^*)^n$$

$$(A^n)^* = (A^*)^n$$





Summary

- Following this topic, you now
 - Understand a number of properties of the adjoint
 - Are aware that these parallel the properties of the complex conjugate





References

- [1] https://en.wikipedia.org/wiki/Hermitian_adjoint





Acknowledgments

None so far.





Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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